Modeling Ultimate Loss-Given-Default and Time-to-Resolution on Corporate Debt

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Abstract

Loss-given-default ("LGD") is a critical parameter in modeling the credit risk of instruments subject to default risk, in addition to various other facets of credit risk modeling. However, another source of uncertainty in addition to LGD is the *time-to-resolution* ("TTR") of the default event, which has been given limited attention in the literature. LGD and TTR are likely to be correlated with each other and both individually to vary significantly on various standard LGD risk factors such collateral characteristics and the macroeconomic environment. As the TTR is often right censored due a cut-off in the data sample underlying the estimator of the LGD, such estimates not accounting for this may suffer from what is known in the statistics literature as censoring, which in the credit risk modeling literature as an LGD *resolution bias,*. Such LGD models not adjusting for this through omitting a consideration of the distribution of TTR, the standard variety prevalent in the industry, will result in biased estimates when applied to non-defaulted performing instruments as the TTR is unknown. In this study we propose to address this issue through the simultaneous modeling of the LGD on resolved cases and TTR on both resolved (non-censored) and unresolved (censored) cases. This study empirically investigates the determinants of LGD and TTR through building alternative predictive econometric models on bonds and loans using an extensive sample of most major U.S. defaults in the period 1985–2022, a dataset consisting of both censored and non-censored observations. The key finding is that when compared with standard approaches that do not account for resolution bias, our approach has superior fit to the data in terms of out-of-sample cases where LGD is unresolved at the point of model development. This study extends prior work by modeling LGD by utilizing an advanced (yet practical to implement) econometric technique, incorporating obligor's complete capital structure characteristics, for a large corporate asset class and rigorously testing the proposed model on an out-of-sample basis for unresolved LGDs.

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1. Introduction

Loss-given-default (LGD), the loss severity on defaulted debt obligations, is a critical component of risk management, pricing, and portfolio models of credit.[2] LGD is among the three primary determinants of credit risk, the other two being *probability-of-default* ("PD") and *exposure-at-default* ("EAD"). However, LGD has not been as extensively studied, and is considered a much greater modeling challenge than PD. Traditional credit models such as PD have focused on systematic components of credit risk that attract risk premiums. Unlike PD, determinants of LGD have typically been ascribed to idiosyncratic, borrower-specific factors. However, there is now an ongoing debate about whether the risk premium on defaulted debt should reflect systematic risk, and in particular, whether the intuition that LGDs would rise in worse states of the world is correct; and how this could be refuted empirically given limited and noisy data. This heightened focus on LGD has been motivated by the large number of defaults and nearly simultaneous decline in recovery values observed through the prior downturns in the credit cycle, as well as what is expected to occur in the current economic slowdown following the distortions of the COVID crisis, the resurgence in inflation and the response of monetary authorities to the latter. We may add to this past banking supervisory responses to credit downturns that lead to developments such as the Basel prudential regulations (Basel Committee of Banking Supervision, 2003, 2017) and the supervisory stress testing exercises (Federal Reserve Board 2009, 2016), and the continued growth in credit markets beyond the traditional domains of supervisory purview (i.e., private credit and Fintech). However, obstacles to better understanding and predicting LGD endure, include a dearth of relevant data in many assets classes and the lack of a coherent theoretical underpinning, a continuing challenge to researchers.

A model for LGD may consider many factors, such the obligor's characteristics (e.g., financial ratios, industry, etc.), collateral types (e.g., seniority rank or collateral type), product features, the macroeconomic environment and the *time-to-resolution* ("TTR"). We define the TTR as the time period where all recoveries have been realized, in the case of private bank loans the point from which there is a determination of unlikeliness to repay or non-payment until all workout recoveries have been deemed collectable ("certified LGDs"), and in the case of the large corporate asset class the resolution of a publicly recognized default event (bankruptcy filing or

[2] LGD is equivalent to 1 minus the recovery rate, dollar recovery as a proportion of par, or exposure at default assuming all debt becomes due at default. We focus on LGD as opposed to recoveries with a view toward credit risk management applications.

distressed exchange) by an event of the entity emerging from default in a reorganization or being liquidated.

In many cases, the length of the data sample used for estimating LGD may be relatively short, while the TTR may be relatively long, resulting in a right censoring of recovery cash flows due to the end date of the sample. Arguably, it is likely that censored observations may have different characteristics from those of non-censored or closed counterparts. Furthermore, if the TTR is correlated with LGD but omitted from the specification of the LGD model, then such model will suffer from a specification error due to both an omitted variable bias as well as a sample selection bias.

This study contributes to the research on LGD in several respects. The methodology constructs internally consistent models of LGD in line with prior empirical findings and theoretical expectations, which has favorable features in line with industry and supervisory expectations regarding model validation (SR11-7, 2011). We extend the prior literature on considering LGD and TTR (Gürtler and Hibbeln 2011; Yashkir and Yashkir 2013; Carey and Gordy 2016; Chen 2018) through using an extensive sample of corporate bond and loan defaults with the consideration of both obligor complete capital structure characteristics and macroeconomic factors. In the process we not only address several academic aspects of LGD, but also offer an actionable approach that may be applies by bank portfolio and risk managers, as well as supervisors and market participants in this space. In the latter domain, distressed debt traders may use this model in forecasting ultimate LGD or as an input into *credit Value-at-Risk* ("C-VaR") models, bank portfolio managers may deploy it in early warning systems, and or bank risk managers or supervisors may consider it for regulatory capital calculations.

Another important aspect of considering TTR in the context of LGD modeling is the implications *stress testing*, which is very important in assessing the credit risk of bank loan portfolios has grown over time. Currently these exercises are accepted as the primary means of supporting capital planning, business strategy and portfolio management decision making (U.K. Financial Services Authority, 2008). Such analysis gives us insight into the likely magnitude of losses in an extreme but plausible economic environment conditional on varied drivers of loss. It follows that such activity enables the computation of unexpected losses that can inform regulatory or economic capital according to Basel III guidance (Basel Committee for Banking Supervision, 2011). In light of this, a model for LGD that explicitly accounts for the time dimension is naturally suited to the development of forward looking forecast scenarios in stress testing exercises. Pineau (2023) proposes a simple approach to include a modelled LGD as part of a stress test using the PD-LGD dependency, studying three such PD-LGD models and obtaining an assessment of capital requirements that depends only on the PD, illustrating the approach by conducting a carbon price stress test on the Stoxx 600 index.

LGD can be defined variously depending upon the institutional setting, the type of instrument (e.g., traded bonds or bank loans), or the credit risk model (e.g., pricing debt instruments subject to the risk of default, expected loss calculation or credit risk capital). The ultimate LGD represents eventual discounted loss per dollar of outstanding balance at default. When considering loans that may not be traded, and taking into consideration when cash was received as well as other losses incurred in the collection process, ultimate LGD is the relevant measure for an input into a regulatory or economic credit capital model. In the case of bonds or marketable loans, one can measure the prices of traded debt at the initial credit event, or the discounted market values of instruments received at the resolution of default. The latter is potentially a proxy for the ultimate LGD, the focus of our study for two purposes. Our primary objective is to provide results of use to agents invested in defaulted securities having time horizons that span the resolution period, who wish to assess expected value upon emergence relative to some benchmark available at default, such as trading or model-based prices. Agents who would benefit include bank workout specialists, risk managers, or vulture hedge fund investors. Second, our results would be relevant for financial institutions attempting to quantify economic LGD for purposes of the Basel Advanced IRB approach to regulatory capital, which requires estimation of the ultimate LGD.

Aspects of this modeling exercise deserving of special attention include the distributional properties of LGD. While the available theory and empirical evidence suggests it to be stochastic and predictable with respect to other variables, in most extant credit models LGD has been treated as either deterministic or as an exogenous stochastic process. The quest for tractability gives rise to such assumptions, but in practical applications this results in understated capital, mispricing, and unrealistic dynamics of model outputs. Our research helps to resolve such deficiencies by modeling ex ante the distribution of LGD as a function of empirical determinants such as contractual features, firm capital structure, borrower characteristics, debt and equity markets variables, and systematic factors. In order to empirically investigate the determinants of, and build predictive econometric models for ultimate LGD we use the Moody's *Ultimate Recovery Database* ("MURD"; Emery et al 2007) an extensive sample of ### defaulted firms (covering 1985 to 2022), a dataset containing the complete capital structures of each obligor. Our sample is highly representative of the U.S. large corporate loss experience over the last two decades.

This paper is organized as follows. In Section 2 we review the literature. In Section 3 we present the econometric models. In Section 4 we present summary statistics and estimation results. Section 5 provides conclusions and directions for future research.

2. Review for the Literature

It may be argued LGD has been a relatively neglected aspect of credit risk research.[3] Starting with the seminal work by Altman [1968], modeling of PD is currently in a relatively mature stage as compared to LGD. The heightened focus on LGD is evidenced by the recent flurry of research into the application of credit models and estimation of LGD from available data. This literature ranges from simple quantification of LGD, to calibration of credit models embedding LGD assumptions, to empirical or vendor models of LGD. The Zeta model of Altman, Haldeman, and Narayanan [1977] was a second generation of the Altman Z-Score PD estimation model. In this model, loan LGD estimates were based on a workout department survey (1971–1975), which yielded conclusions regarding the magnitude of discounted post-restructuring recoveries on unsecured bank loans. Bank studies focusing on internal loan data included JP Morgan Chase (Araten, Jacobs, Jr., & Varshney [2004]), where the authors studied ultimate workout LGD for wholesale loans during 1982–2000.

Among early studies relying almost exclusively on secondary market prices of bonds or loans soon after the time of default, Altman and Kishore [1996] estimated LGDs for defaulted senior secured and senior unsecured bonds from 1978–1995, yielding estimates that could be statistically distinguished among various industry groups. Altman and Eberhart [1994] and Fridson, Garman, and Okashima [2000] provided evidence that the more senior bonds significantly outperformed the more junior bonds in the post-default period, results confirmed by Hamilton, Gupton, and Berthault [2001] for secondary market loan prices a month after default.

[3] Acharya, Bharath, and Srinivasan [2007]; Altman, Resti, and Sironi [2001, 2003]; Altman and Fanjul [2004]; Altman and Ramayanam [2006]; Araten, Jacobs, and Varshney [2004]; Carey and Gordy [2007]; Frye [2000a, 2000b, 2000c]; and Jarrow [2001].

Emery [2003] and Altman and Fanjul [2004] compared LGDs on bank loans and bonds, respectively, as inferred from the prices of the traded instruments at default in a Moody's database, revealing that loans experience lower loss severities when controlling for seniority. Cantor, Hamilton, and Varma [2003] showed similar findings for corporate bonds as Altman and Fanjul, and additionally found differential LGD by rating at origination, such that "fallen angels" of the same seniority had significantly lower LGDs.[4]

Among studies that looked at ultimate LGD, Standard and Poors (Keisman & van de Castle [2000]) presented empirical results from the LossStats™ in the period 1987–1996 for marketable bonds and loans. This study also showed that the influence of position in the capital structure (i.e., the proportion of debt above or below a claimant in bankruptcy) was independent of collateral and seniority in determining loss severities. A more recent rating agency study by Moody's (Cantor & Varma [2004]) examined the determinants of ultimate LGD for North American corporate issuers over a period of 21 years (1983–2003), looking at many of the variables considered herein (e.g., seniority and security; firm, industry-specific, and macroeconomic factors).

Several recent empirical studies of LGD attempting to measure LGD–PD correlation have put more structure around this exercise, either by building predictive econometric models, or by attempting to directly test models. Frye [2000a, 2000b, 2000c] examined the LGD–PD correlation in extensions of the Merton [1974] framework allowing for systematic recovery risk, finding a significant negative relationship at various levels of aggregation. Other studies examined this by looking at LGD as implied from the prices of traded debt at or prior to default, as opposed to ultimate LGD, or the "reduced form" approach. Among these, Jarrow [2001] developed a hybrid structural–reduced model, in which PDs and LGDs were functions of the underlying state of the macro-economy. Hu and Perraudin [2002] also examined this relationship and found LGD–PD correlations on the order of 20%. Jokivuolle and Peura [2003] presented an option theoretic model for bank loans and were able to produce a positive correlation between PD and LGD. Bakshi et al. [2001] extended the reduced-form approach by allowing a flexible correlation between PD, LGD, and the risk-free rate. Imposing a negative correlation between PD and LGD, they found that a 4% increase in the (risk-neutral) hazard rate was associated with a 1% increase in the expected LGD. In related work on the resolution of

[4] Median LGDs of 49.5% and 66.5% for defaulted issuers originally investment and speculative grade, respectively.

default focusing on high-yield debt portfolios, Parnes [2009] developed a theoretical model that explicitly incorporated LGD assumptions.

Several studies showed that LGDs tend to rise more in periods of recession than they fall during expansions, suggesting that more is at play than a macroeconomic factor influencing the value of collateral. Keisman and van de Castle [2000] found that during the earlier stress period at the beginning of the previous decade, LGDs of all seniorities rose in the S&P LossStats™ database for the period 1982–1999. Altman, Resti, and Sironi [2001, 2003] also found that LGDs increase as the credit cycle worsens and as default rates increase above the cycle's long-run average.[5] Araten et al. [2004] related unsecured U.S. large corporate borrower-level LGDs to the average Moody's All-Corporate default rate and reported similar behavior. However, Altman, Brooks, Resti, and Sironi [2005] found that a systematic variable had no effect on LGD when controlling for bond market conditions (e.g., supply–demand imbalances). Acharya, Bharath, and Srinivasan [2007] examined the same data and time period as Keisman and van de Castle [2000], and while they verified that seniority and security are key determinants of LGD, in addition they found industry-specific factors influencing LGD independently of the macroeconomic state and bond market conditions analyzed in Altman et al. [2005]. In particular, after controlling for firm-specific, contractual, and systematic factors they found elevated LGDs in distressed industries, defined as those sectors having significantly lower profitability than the overall market. They argued that in these cases fewer redeployable assets, greater leverage, and lower liquidity is driving lower average recoveries; and that their results support a test of the Schleifer and Vishny [1992] "fire-sale" hypothesis, an industry equilibrium phenomenon in which macro and bond market variables are spuriously significant due to omitting an industry factor.

Serval studies argued for a two-stage approach to measuring LGD, first estimating an "estate LGD" at the obligor level, and then treating instrument-level LGDs according to a contingent claims approach, as under APR such recoveries can be viewed as collar options on residual value of the firm. However, it has been argued that the endogeneity of the bankruptcy decision would result in a measurement problem in the first-stage borrower level, and furthermore an extensive literature on violations of APR suggested a similar problem in the second-stage instrument level (Eberhart, Moore, & Rosenfeldt [1989]; Hotchkiss [1993]; Weiss

[5] Altman [2006] reported that his model overpredicts LGD in recent years, which he speculated may be due to bubble conditions in the high-yield market.

[1990], Carey and Gordy [2007, 2016]). Jacobs and Karagozoglu (2001) follows in this line of research by building a simultaneous equation model in the *beta-link generalized linear model* ("BLGLM") class for an extensive sample of 871 defaulted firms in the period 1985-2008 from the MURD dataset used herein, containing the complete capital structures of each obligor. In a departure from the then extant literature the authors find the economic and statistical significance of firm-specific, debt, and equity-market variables, and a then new finding that larger firms have significantly lower LGDs while larger loans have higher LGDs.

Several empirical LGD studies have shown that TTR can play an important role in LGD recoveries (see, for example, Gürtler and Hibbeln 2011;Yashkir and Yashkir 2013; Carey and Gordy 2016). Chen (2018), one of the key studies that we are extending, addresses this topic through a novel approach modeling LGS and TTR simultaneously issue, adopting the fractional response logit model of Papke and Wooldridge (1996). The author empirically demonstrates the applicability of this model with a banking workout LGD dataset having both censored and non-censored LGSs for five model specifications, finding that as compared with a mores standard approach there is superior performance, as well as a more conservative prediction of LGD and greater marginal sensitivity to macroeconomic factors. An advantage of this approach that explains these favorable results is that the model of Papke and Wooldridge (1996) predicts a LGD with bounded domain of between 0 and 1 (Li et al 2016), yet tends to outperform more complex models such as linear models with LGD (or log-odds transformed LGD as the dependent variable, other alternatives include logistic growth, censored Tobit, zero-inflated beta models, decision trees (Grunert and Weber 2009; Bastos 2010; Qi and Zhao 2011; Yashkir and Yashkir 2013); as well as the BLGLM model of Jacobs and Karagozoglu (2011), which is applied to an earlier version of the MURD dataset that is used in this study. That said, these studies and others find that for any given dataset, model performance is mainly driven by the proper choice of risk factor covariates to model LGD.

We also make note of this evidence regarding the PD–LGD correlation influencing the Basel II guidelines: paragraph 468 on downturn LGD in the Bank for International Settlements (BIS) Accord (Basel Committee of Banking Supervision [2003, 2004]), and the additional guidance offered by the BIS (Basel Committee of Banking Supervision [2005]). Basel II requires advanced internal ratings based (IRB) banks to not only capture all relevant risks regarding possible cyclical variability in LGD, but at the same time states that bank estimates of long-term ultimate LGDs having no such systematic variations may be acceptable. Miu and Ozdemir [2006] argued that banks can incorporate conservatism into cyclical LGDs estimated in a point-in-time framework, without an LGD–PD correlation; however, they estimated commensurate increases in credit capital to compensate for this.

Finally, we will mention the stress testing context in in this domain, n satisfaction of the *Current Expected Credit Loss* ("CECL"; Financial Accounting Standards Board, 2016) accounting standards or compliance with the Federal Reserve's *Dodd-Frank Stress Testing* ("DFAST"; Board of Governors of the Federal Reserve System – "FRB", 2016) program, we observe that the predominant types of models used in the industry differ slightly from the earlier context of the financial crisis, as the applications must meet particular capital adequacy and accounting requirements not previously a consideration (Global Credit Data, 2019; "GCD"), which speaks to the importance of loan level modeling which is naturally the level at which LGD analytics is performed. Dwyer et al (2014) develops a model that forecasting losses under stressed scenarios is to model losses by combining separate PD, LGD, and EAD models that can serve as the LGD component of such a model suite. Their stressed LGD model projects recovery rates based upon stressed macroeconomic variable scenarios, debt type, sector and stressed PD levels. The model uses macroeconomic variables from the Federal Reserve Board's CCAR scenarios to facilitate the use for CCAR post-stress capital analysis.

Our approach extends the existing research along several dimensions. First, we contribute to prior work by modeling LGD jointly with the TTR at the instrument level. In particular, a simultaneous equation estimation of LGD and TTR is advantageous in that we can model address the aforementioned issues resolution bias and censoring coherently. Second, as compared to extant work in the stream of research addressing LGD and TTR, using the MURD database we integrate new variables such as contractual, capital structure factors and obligor specific financial ratios, and address the large corporate asset class not previously considered in this line of research. Finally, in addition to these we confirm many of the findings of the literature in regard to determinants of LGD such as industry characteristics, and macroeconomic considerations.

3. Econometric Modeling of LGD with Duration Dependence

A useful taxonomy to characterize *stochastic data generating processes* ("SDGPs") consists of four dimensions (Chern, 2018) according to whether the state and time are discrete or continuous. An example of a process in credit risk management where both state and time are discrete is modeling account delinquency status (e.g., if 60 or 90 days past due is true or false) as measured over fixed time intervals (e.g., monthly, quarterly or annually.) Typically, *binary choice models* (e.g., logistic regression) are used to model the probability of account delinquency or default in a fixed time interval. In the case where time is continuous and the state is discrete, commonly *survival models* are applied to predict the of time to event, examples kindred to default and recovery credit risk applications being analysis of duration of unemployment duration in labor economics or longevity in population demographics. In the case of continuous response variable where time is a fixed intervals we see the application of regression models such as OLS or time series, examples in credit risk including spending or balance analysis within predefined interval. We could classify in this category standard models of LGD that do not account for TTR models that ignore time to recovery and are thus subject to model specification error. Finally, where we have both continuous state and time that are not predetermined is where we can classify the LGD model considered herein, having a with its variable performance duration window of TTR. In this case a measurement of LGD is considered to have occurred when the all the recoveries have been received, the last workout cash flow in the case of bank loans, else the instruments received in settlement or reorganization flowing a default in the case of traded loans or bonds. This is in contrast to LGD mortality rate analysis which requires accurate measurements of the recovery discounted cash flows for all time periods (Dermine and Neto de Carvalho 2006). As the task herein is forecasting of LGD starting at the default date, we need to not convert the continuous TTR into discrete intervals as in the mortality rate analysis, which implies that the cumulative recovery workout recoveries need only be observable at the end of the workout period. In the case of the URD database of loans and bonds, the recovery value is represented by the prices of the debt at the resolution of the default event (so that the LGD 1 one minus this amount as a percent of par value).

The latent variable system of equations for LGD and TTR may be written as

$$LGD^{*}=\begin{cases} LGD & for\ resolved\ instances, \\ missing & for\ unresolved\ instances, \end{cases} \tag{1}$$

and

$$t^{*}=\begin{cases} t & for\ resolved\ instances, \\ \bar{t} & for\ unresolved\ instances, \end{cases} \tag{2}$$

where LGD^* is the latent variable for LGD^*. We may represent the dependence of LGD upon dependence on TTR in the following form,

$$LGD^* = \Lambda\left(\boldsymbol{\beta}^T \mathbf{x} + \delta t\right) + \upsilon, \tag{3}$$

where $\mathbf{x}$ is a vector of explanatory variables as of the default date, $\boldsymbol{\beta}$ is a vector of coefficients, $\upsilon \sim g\left(\bullet\right)$ is a random error term and t is the latent TTR variable (with coefficient δ) where we admit that the latter may covary with the LGD. $\overline{t}$ represents the TTR as of the LGD sample end date for the censored episodes. In equation (1) the LGD will be missing if the default has not been resolved or the TTR is right censored. Following a property of the linear exponential family of distributions, the model in equation (3) can be consistently estimated using a fractional response (Papke and Wooldridge, 1996), where $\Lambda\left(\mathbf{x},t\right) \in \left(0,1\right)$ for the set of explanatory variables $\mathbf{x}$ and the TTR $t \in \mathbb{R}^+$ assuming that the mean LGD $\Lambda\left(\mathbf{x},t\right)$ is correctly specified. Alternatively, we may estimate this model as a *nonlinear logistic growth* specification model by further assuming the correct error distribution for $\upsilon \sim g\left(\bullet\right)$ jointly with an orthogonality condition $E\left(\upsilon \middle| \mathbf{x},t\right) = 0$.. If TTR is correlated with LGD but omitted from the above model specification in (3), the LGD model developed on the historical defaulted loans will be biased when applied to non-defaulted obligations (Greene, 2012). Furthermore, we may enforce the positivity of the TTR and $E\left(t \middle| \mathbf{z},\boldsymbol{\gamma}\right) > 0$ by modeling the TTR as a logarithmic transformation

$$\mathrm{Log}\left(t\right) = \boldsymbol{\gamma}^T \mathbf{z} + u, \tag{4}$$

where $\mathbf{z}$ is a vector of explanatory variables as of the default, $\boldsymbol{\gamma}$ is a vector of coefficients and υ is a random error term following a distribution function $u \sim f\left(\bullet\right)$. Commonly used parametric distributions for the TTR include gamma, lognormal, log-logistic, exponential and Weibull distribution (Allison 2007), all of which we consider as part of the model development process. It is further assumed that with sufficient time the instance of default will be resolved, which may be a reorganization or liquidation for loans or bonds, so that the missing LGD resulting from the right censoring of TTR will become eventually become observable if the data sample period can be extended beyond the current sample end date.

In order to estimate the selection equation governing the TTR, a right censored log-likelihood function can be maximized using both resolved and unresolved observations, denoted as

$$\text{LogL}^{ttr}\left(t,\mathbf{z},\boldsymbol{\gamma}\right)=\sum_{resolved}\log\left[f\left(u\middle|t,\mathbf{z},\boldsymbol{\gamma}\right)\right]+\sum_{unresolved}\log\left[1-F\left(\overline{t}\right)\right], \tag{5}$$

where $F\left(\overline{t}\right)$ is the *cumulative density function* ("CDF") of u evaluated at $\text{Log}\left(\overline{t}\right)-\boldsymbol{\gamma}^T\mathbf{z}$ in the case of unresolved default. As there is a relationship of recursion between equations (3) and (5), coupled with conditioning upon right censoring of the final date in the LGD data sample, the model of LGD in equation incorporating TTR for resolved default may be estimated by maximizing the following log-likelihood function with explanatory variables $\{\mathbf{x},t\}$ and random disturbance term υ:

$$\begin{aligned}\text{LogL}^{\text{lgd}}\left(LGD,\boldsymbol{\beta}^T\mathbf{x},\delta t\right)&=\sum_{resolved}\log\left[g\left(\upsilon\middle|LGD,\boldsymbol{\beta}^T\mathbf{x},\delta t\right)\right]\\&=\sum_{resolved}\log\left[g\left(LGD-\Lambda\left(\boldsymbol{\beta}^T\mathbf{x}+\delta t\right)\right)\right],\\&=\sum_{resolved}\log\left[g\left(\upsilon\middle|LGD-\frac{1}{\exp\left(-\left(\boldsymbol{\beta}^T\mathbf{x}-\delta t\right)\right)}\right)\right],\end{aligned} \tag{6}$$

where we choose the unit interval logistic growth model functional form,

$$\Lambda\left(\boldsymbol{\beta}^T\mathbf{x}+\delta t\right)=\frac{1}{\exp\left(-\left(\boldsymbol{\beta}^T\mathbf{x}-\delta t\right)\right)}\in\left[0,1\right]. \tag{7}$$

As an alternative to the above derivation, we may assume that the LGD is generated from a distribution in the *linear exponential family*, which means that it inherits a log-likelihood function of the Bernoulli type. In this setting, *quasi-maximum likelihood* estimation may be deployed to estimate the fractional response logit model in the following manner (Papke and Wooldridge, 1996) as follows:

$$\text{LogL}^{\text{lgd}}\left(LGD,\boldsymbol{\beta}^T\mathbf{x},\delta t\right)=\sum_{resolved}LGD\times\log\left[\Lambda\left(\boldsymbol{\beta}^T\mathbf{x}+\delta t\right)\right]+\left(1-LGD\right)\times\log\left[1-\Lambda\left(\boldsymbol{\beta}^T\mathbf{x}+\delta t\right)\right]. \tag{8}$$

In either case of the logistic growth model (7) or the fractional response logit model (8), the expected LGD conditional on the risk factors $\{\mathbf{x},t\}$ is given by

$$\mathrm{E}\left(LGD, t, \boldsymbol{\beta}^T \mathbf{x}, \boldsymbol{\gamma} T\mathbf{z}, \delta\right) = \Lambda\left(\boldsymbol{\beta}^T \mathbf{x} + \delta t\right) = \frac{1}{\exp\left(-\left(\boldsymbol{\beta}^T \mathbf{x} - \delta t\right)\right)}. \tag{9}$$

Whether the default instance is resolved or unresolved, the complete log-likelihood function $\mathrm{LogL}^{\mathrm{lgd}}\left(LGD, t, \boldsymbol{\beta}^T \mathbf{x}, \boldsymbol{\gamma}^T \mathbf{z}, \delta\right)$ conditional upon the data sample $\{LGD, t, \mathbf{x}, \mathbf{z}\}$ is given by the summation of the marginal log-likelihood function $\mathrm{LogL}^{\mathrm{ttr}}\left(t, \boldsymbol{\gamma}^T \mathbf{z}\right)$ from the TTR equation (5) and $\mathrm{LogL}^{\mathrm{lgd}}\left(LGD, \boldsymbol{\beta}^T \mathbf{x}, \delta t\right)$ the LGD equation, conditional upon t:

logLt2r.t; z□/ for time to recovery t (3.3a) and logLlgd.LGD; xˇ; tı/ forLGD(3.3b),
conditional on t :
log L.LGD; t; xˇ; z□; ı/
D
X
closed
logfg.v j t/g C
X
closed
logff .t/g C
X
open
logf1 □ F.Nt/g
D
X
closed
logfh.u; v/g C
X
open
logf1 □ F.Nt/g; (3.4)
where the joint PDF is h.v; u/ D g.v j u/f .u/, the product of marginal distribution
for time to recovery and conditional distribution for LGD given time to recovery.
Therefore, due to the recursive relationship between LGD and time to recovery, the
estimation of the complete loglikelihood function (3.4) can be achieved by the separate

estimations of time to recovery model (3.3a) and the LGD model (3.3b) or (3.3b0).

Since time to recovery t is unknown before the default starts, the expected LGD (conditional on the observed covariates x and z for credit risk management) can be obtained by integrating out t :

E.LGD j x; z/ D Et fEujt .LGD j x; t/g D

Z 1

0

□.xˇ C ıt/f .t j z□/ dt: (3.5)

A consistent estimation of E.LGD j x; z/ can then be obtained by substituting the consistent parameter estimates Oˇ and O ı into □.x Oˇ C ıOt / as well as □O into f .t j z□O /

in the above equation.

Further, for credit risk management applications (such as stress testing and economic policy analysis) the LGD's sensitivity to marginal changes in covariate

m 2 fx; zg is given by

@E.LGD j x; z/

@m

D

Z 1

0

□

@f□.xˇ C ı/g

@m

f .t j z□/ C □.xˇ C ıt/

@f .t j z□/

@m

□

dt: (3.6)

As shown above, (3.6) can again be estimated by replacing the parameters ˇ, ı and □

with the consistent estimates O˘, ıO and □O , respectively.

3.3 Model estimation procedure

The expected LGD in (3.5) can be obtained in three steps. The first step entails estimating a traditional LGD model, that is, (3.3b) or (3.3b0), and including time to recovery in the regression, thus addressing the omitted variable bias. The statistical significance of the time to recovery variable can then be evaluated for the correlation between LGD and time to recovery. The prediction bLGD.x; t / is obtained for both closed and censored (or open) transactions. In the second step, the time to recovery survival model (3.3a) must be estimated for the probability distribution of time to recovery bPr.z/ using both closed and open transactions in order to avoid the sample selectivity problem. For open transactions, the right censored sample end date is used to calculate the truncated time to recovery N t in the model specification. Finally, based on the predicted bLGD and the predicted probability bPr from the above two steps, the expected LGD conditional on covariates .x; z/, E.LGD j x; z/, may be computed for production implementation for performing loans by eliminating the unobserved time to recovery as follows:

bLGD.x; z/ D

XT

tD1

fbPr.t j z/

bLGD.t j x/g; (3.7)

where the summation of time to recovery runs from month one to month T to ensure that the probability beyond the upper bound T is negligible in the evaluation.

Note that this approach does not conceptually require an upper bound for time to recovery, although empirically all banks would like it to be as short as possible. Using this approach, the final expected LGD in (3.5) can be consistently estimated, provided the LGD and time to recovery models are correctly specified with the consistent parameter estimates O˘, ıO and □O . The LGDsensitivity in (3.6) can be similarly obtained

by evaluating the estimated LGD for different values of the covariate m.

Data and Summary Statistics

We have built a database of defaulted firms representative of the U.S. large corporate loss experience (bankruptcies and out-of-court settlements), all having Moody's rated instruments at some point prior to default. Our database is constructed through merging the December 2008 release of the Moody's Ultimate Recovery Database™ (MURD) with information from various sources such as www.Bankruptcy.com, Edgar SEC filings, LexisNexis, Bloomberg, Compustat, and the Center for Research in Security Prices (CRSP). It contains data on 3,902 defaulted instruments from 1986 to 2006 for 871 borrowers, for which there is information on all classes of debt in the capital structure at the time of default. Despite our sample selection being driven mainly by data availability in the matching of the MURD database to other databases, the final dataset is largely representative of the U.S. large-corporate default experience over the last 20 years.[6] All instruments are detailed by debt type, seniority, collateral type, position in the capital structure, original and defaulted amount, resolution outcome, and instrument price or value of securities at the resolution of default (emergence from Chapter 11 bankruptcy as an independent entity or acquisition by a third party, Chapter 7 liquidation or out-of-court settlement of a distressed exchange). It includes either the prices of prepetition instruments at the time of emergence from, or prices of new instruments received in settlement of, bankruptcy or other distressed restructuring, respectively. For a subset of observations, we can obtain the prices of traded debt, the equity prices, or financial statement data at around the time of default.[7] We calculate economic LGD by discounting nominal LGD by the coupon rate on the instrument prevailing just prior to default.[8]

Exhibit 1 presents the counts and sample averages of various key variables: trading instrument inferred LGD at default, discounted LGD, number of major creditor classes, principal at default, and time to final resolution. These are presented at the instrument level and the obligor level, and then further broken down among bankruptcies and out-of-court settlements. Most defaults are bankruptcies as opposed to out-of-court settlements, with 3,273 bankruptcies versus 629 settlements at the instrument level, and 728 bankruptcies versus 143 out-of-court settlements at the obligor level.

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[6] A statistical analysis, available upon request, shows that the final datasets used in the regression analyses are demographically similar to the broader MURD database.

[7] In the case of debt, we take the average price from the 15th to the 45th day after default from the Moody's Default Risk Service (DRS™) database. For financials and equity prices, we look to COMPUSTAT or CRSP in either the nearest quarterly filing date or month prior to default, respectively.

[8] We also replicate results with a risk-free term structure, as in Carey and Gordy [2007], as well as using a high yield index, as in Acharya et al. [2007], and results are not materially different.

Our dataset contains the prices of traded debt available at the time of default for only 1,118 out of 3,902 (or 28.6%) of the instruments and for 460 out of 871 (or 52.8%) of the obligors. Average LGDs across the entire sample as inferred from the prices of traded instruments at default are significantly higher than ultimate LGDs, 61.04% versus 43.21% at the instrument level, and 63.46% versus 46.88% at the obligor level, consistent with previous research (Cantor, Emery, & Keisman [2007]).[9] Discounted LGD is much higher for bankruptcies as compared with out-of-court settlements, 48.38% versus 16.31% at the instrument level, and 51.41% versus 23.80% at the obligor levels[10]. We also see the marked non-normality of the LGD estimates, which is accentuated for the ultimate LGDs as compared to the trading prices, with standard deviations at the instrument level of 40.4% versus 28.7% and at the obligor level of 31.8% versus 23.9%.

Firms in the database tend to have about two major creditor classes in a range from 1 to 6, with an average of 2.44 at the instrument level and 2.20 at the obligor level. Furthermore, and perhaps surprisingly, if we take this as a measure of the complexity of the capital structure, out-of-court settlements do not tend to have fewer creditor classes than bankruptcies, an average of 2.55 versus 2.42 at the instrument level, and 2.31 versus 2.18 at the obligor level. Average time to resolution from first instrument default to final instrument resolution is 1.38 years at the instrument level and 1.36 years at the obligor level. This is a long-tailed distribution, ranging from a day in some of these prepackaged bankruptcies or out-of-court settlements, to a 9.18-year bankruptcy resolution process. Out-of-court settlements resolve much more quickly, on average 0.21 years for the instrument level and 0.42 years for the obligor, and the range of the distribution is compressed, ranging from 0.002 to 5.64 years.

Exhibit 2 presents distributional statistics and spearman rank order correlations with LGD (both at the obligor and instrument levels) for selected variables in our database. We highlight qualitatively a few key statistics here that have economic meaning or relate to our final results, first considering borrower financial ratios. LGD is negatively correlated with leverage measures, a strongest effect for the total debt to market value of equity ratio as compared to the ratio of total debt to book value. Variables measuring size, like book value of assets or market value of

[9] The finding that while market implied LGD has predictive power for ultimate LGD, it is an upwardly biased estimator, which has been ascribed to either the illiquidity of defaulted debt markets, investors' extreme risk aversion, or even an over-shooting effect.
[10] This observation is consistent with the predictions of several theoretical models of the resolution of default (e.g., Parnes [2009]).

equity, correlate with LGD positively at the firm level and negatively at the instrument level. Tobin's Q, a measure of the degree to which there may be unrecorded firm value, has relatively strong positive correlation with LGD at both obligor and instrument levels. The intangibles ratio (IR), which conveys similar information, also has positive correlations with LGD that are of greater magnitude at the obligor level. Variables measuring cash flow (working capital to total assets or operating cash flow ratios) are both negatively correlated with LGD at mild to reasonably strong levels. Measures of industry profitability (e.g., return on assets or profit margin) are moderately and negatively correlated with LGD.

Among variables representing capital structure characteristics, number of instruments, and number of creditor classes, the degree to which the firm's debt is secured or the degree to which banks are represented in the creditor group is positively correlated with LGD. Considering variables measuring the amount of credit risk, the best predictor of ultimate LGD is the loss given default from the price of traded debt at default, having a very strong positive correlation. Cumulative abnormal returns (CAR)[11] on equity prices in the 90 days until default (median of -12%) have the second strongest positive correlation to LGD. The number of downgrades prior to default is negatively correlated with the LGD, which could imply either that LGD is lower if obligors were originally more highly rated, or if the migration to default is more gradual. Correlations of variables capturing contractual features show that instruments having subordinated seniority ranks, inferior collateral quality segments, or thinner debt cushions have significantly higher LGDs, while the debt cushion measures have similarly strong relationships.

In the set of variables measuring time spans of interest, we see rather strong relationships: time to maturity of defaulted instruments is positively related to discounted LGD, and time between defaults is negatively related to discounted LGD. Finally, we observe evidence that LGDs are elevated during downturns, as LGD is increasing (decreasing) in the Moody's speculative grade default rate (S&P 500 equity index return), consistent with the large body of empirical evidence as discussed previously in the literature review.

Empirical Results

[11] We calculate 12-month CAR at each date in the 90 days prior to default using the CRSP NYSE/AMEX/NASDAQ equally weighted market index as the benchmark.

In this section we discuss our empirical findings for the full information maximum likelihood estimation (FIMLE) of the beta link generalized linear model (BLGLM), a simultaneous equation analysis of instrument and obligor LGD. Exhibit 3 presents results from the FIMLE of the multivariate BLGLM model. FIMLE is used to model the endogeneity of the relationship between LGD at the obligor (firm) and instrument levels in an internally consistent manner. This technique enables us to build a model that can help us understand some of the structural determinants of LGD, and potentially improve our forecasts of LGD. All variables are economically significant, as well as statistically significant, in all cases at conventional confidence levels.

We find that the financial statement and capital structure variables should only appear in the equation for obligor LGD, while the contractual feature variables should only appear in the instrument equation. There is a clear economic argument for this in that fundamental factors should influence the firm-level recoveries, while structure of the loan determines to what extent the instrument is "in the money." We also find that CARs and obligor level LGD at default best belong in the obligor equation, and that instrument LGD at default and ultimate obligor LGD best belong in the instrument equation. The latter variable, ultimate obligor LGD in the ultimate instrument LGD equation, captures the feedback loop between the two dependent variables. In the macroeconomic category, all the variables are found to belong in the obligor equation. However, in some other categories we find no clear division, in that some variables were best excluded from one equation whereas others were best included in both within the same category. One case is the credit quality/market category, where investment grade at origination appears in the obligor equation, while principal at default appears only in the instrument equation. Another case is the vintage category, where time between defaults and time to maturity appear only in the obligor and instrument equations, respectively.

Of the credit quality/market variables, the two most important determinants of the ultimate instrument LGD are the instrument LGD at default and the ultimate obligor LGD. To capture the feedback between the two levels of loss severity, we estimate partial effects of 0.21 and 0.19, respectively, which implies that for a 10% increase in either of these we can expect ultimate instrument LGD to increase by about 2%. This says that modeling instrument level recovery akin to a collar option on firm enterprise value does not give the whole story, as there is information about this also embedded in the distressed debt markets. In the obligor equation, for

CAR we estimate a partial effect of -0.28 (i.e., if CARs are 30% higher, then ultimate obligor LGD is expected to be about 10% lower), so that there is additional information on the ultimate recovery encoded in the equity market. Our interpretation is that as the market makes a directionally correct assessment of future expected recoveries, trading patterns result that give rise to this price pattern: Arbitrageurs could be going long or short credit on the perceived better or worse equity of recovering issues from otherwise matched issuers nearing default.[12] Principal at default is slightly less economically significant but no less meaningful, having a partial effect of about 0.001 in the instrument equation (i.e., as the loan amount increases by a factor of 10, LGD decreases by about 10%). This is interesting in that it confirms the hypothesis that loan size may be proxy for unobserved LGD determinants such as complexity of the workout process or coordination issues among creditors. In the context of Basel II, this speaks to the correlation between LGD and EAD, as evidence that post-default recovery risk may be positively correlated with exposure risk prior to default. Finally, the dummy for investment grade (at earliest rating date or origination), which only appears in the obligor equation, has a partial effect of -0.07. All else being equal, credits originally rated as investment grade have 7% lower estate level LGD, which is consistent with the well-known empirical finding that "fallen-angels" have lower loss severities (Altman & Ramayanam [2006]).

Among the set of variables measuring the effect of the business cycle, we observe that the dummy variables for the 2000–2002 and 1989–1991 recessionary episodes enter the obligor equation with partial effects of 0.17 and 0.11, respectively, so that LGD was on this order (17% and 11%) during these periods even after controlling for other systematic factors. This is evidence of *contagion* effects at play, or the phenomenon that there is more LGD risk than can be explained by observable factors due to either unobserved heterogeneity or clustering of defaults during stress periods (Lando & Nielson [2010]). The two other macro variables, Moody's speculative default rate and the S&P 500 return, have partial effects of 0.07 and 0.14, respectively. Therefore, a doubling of default rates increases LGD by about 14%, while a 10% increase in the stock market reduces LGD by about 1%.

[12] An alternative explanation involves the empirical asset pricing results of Vassalou and Xing [2004], who found that the Fama and French [1992] size and value premia were concentrated among the companies having the highest default risk. If we assume a negative relationship between recovery risk and default risk (as in standard versions of the Merton [1974] structural model of credit risk), then it may be that larger and more highly valued companies nearing default would have lower premia and higher prices leading up to default, as well as higher than average ultimate recoveries. This is because our CARs are measured prior to default, whereas the excess returns of Vassalou and Xing measured this ex ante according to the default risk indicator (DIA) that they developed.

In the case of the variables measuring the legal setting, the economic impact of bankruptcy filing is measured by a partial effect of 0.14, so that firms entering Chapter 11 have 14% higher LGDs than those who default and renegotiate their debt. On the other hand, the indicator for a pre-packaged bankruptcy has a partial effect of -0.04, which means that these firms have 4% lower LGD on average. These results are in line with much of the literature on the resolution mechanisms and outcomes with respect to default or financial distress (Jacobs et al. [2010]).

Now let us consider the financial variables appearing only in the obligor equation. The scale variable book value is statistically significant with a negative sign on the coefficient estimate, a partial effect of -0.08, so that firms an order of magnitude larger by this measure have 8% lower ultimate LGD. It may be argued that a larger firm may have the wherewithal to more successfully navigate a default and be rehabilitated due to various factors associated with size (e.g., market power, government support), which is associated with superior recoveries (Jacobs et al. [2010]). The leverage variable debt-to-equity ratio has considerable economic impact, having a partial effect of -0.09, so that a doubling of leverage decreases loss severity by about 20%. At first, this result might seem counterintuitive, since leverage measures are closely related to probability of default (PD) in the Merton [1974] structural modeling framework. However, in the case of LGD this logic may be reversed (e.g., an association between leverage and incidence of default due to financial distress as opposed to insolvency, which all else equal implies higher recoveries). In the context of creditors foreclosing upon the firm, an optimal boundary may be elevated for more leveraged firms, resulting in mitigated recovery risk (Carey & Gordy [2007]). Tobin's Q exhibits economic significance of about the same order as the leverage ratio, having a partial effect of 0.073. The intuition here for a positive relationship with LGD is that investors in these far less than efficient markets are grossly overvaluing these firms prior to default, and in spite of whatever mechanism gives rise to such speculative froth (e.g., informational asymmetries), this is reversed at post default as recovery uncertainty is reduced. The intangibles ratio achieves economic significance at a partial effect of 0.098, implying that a doubling of this ratio results in about a 20% higher LGD. This sign is expected based upon prior research suggesting that defaulted firms whose defaulted assets are of a more tangible quality (e.g., utilities or heavy manufacturing) are more likely to generate better recoveries (Altman & Kishore [1996]). Operating cash flow exhibits considerable economic significance as well, having a partial effect of $-8.31X10^{-3}$, in line with our expectation that firms with a business that is fundamentally more viable are better able to rehabilitate through the resolution process, resulting in greater recovery cash flows and lower ultimate LGDs. Finally for this group, the liquidity measure of working capital to total assets has a partial effect of -0.13. This may be similarly rationalized in that relatively robust cash flow or profitability—a measure of underlying viability of the defaulted obligor's business prospects—could imply a more successful resolution process and superior recoveries (Jacobs et al. [2010]).

Considering industry specific factors, we first see that industry profit margin has a partial effect of -0.092, consistent with the Acharya et al. [2007] fire-sale effect. The dummy variable for the technology has partial effects of 0.06 at the instrument level and 0.03 at the obligor level. The dummy for the utility industry, appearing only in the obligor equation, has a partial effect of -0.15. The latter results, which hold above and beyond the macroeconomic covariates appearing in the regression, are further evidence of the independent importance of industry specific factors in explaining LGD and the plausibility of a separate systematic factor governing LGD apart from that driving default (Frye [2000a, 2000b, 2000c]).

The results for the contractual loan structure variables appearing only in the instrument equation tell a similar story as the financial variables just discussed, both in terms of quality of estimates and congruence of signs on coefficients with our prior expectations. The dummy variables for creditor class, with the base class being bank debt, are all of expected sign and all statistically significant, with partial effects of 0.04, 0.07, 0.23, and 0.11 for senior secured, senior unsecured, senior subordinated, and junior subordinated, respectively. The results for collateral rank are also robust in terms of economic and statistical significance, with a partial effect of 0.15. This reflects the previous findings and intuition that more senior and better-secured instruments have better recoveries (Altman [2006]). Finally for this group, the coefficient estimates on proportion of debt above and below in the capital structure are in line with expectations and precisely measured, with respective partial effects of 0.12 and -0.29. The strength of the debt cushion measures is consistent with theoretical predictions (Carey & Gordy [2007]) and empirical findings (Keisman & van de Castle [2000]).

Addressing the firm capital structure variables appearing only in the obligor equation, note that results are in line with previous results and robust in economic and statistical significance. The number of creditor classes is significantly and inversely associated with the ultimate LGD, with a partial effect of -0.08. This is intuitive if the number of instruments is viewed as a proxy for the capital structure complexity of a firm, which we would expect to be associated with a more difficult or drawn-out resolution process, and consequently a higher economic LGD. Similarly, the proportion of secured debt is economic and statistically significant, having a partial effect of -0.13. Lastly for this group, the proportion of bank debt in the capital structure is also economically and statistically significant, with a partial effect of -0.23, also consistent with intuition regarding the power of the firm's secured or senior creditors.

The final set of variables that we discuss measure time periods of relevance, the (maximum) time between instrument defaults appearing in the obligor equation, as well as the time to maturity at the time of default appearing in only the instrument equation. The time between defaults is notably weaker in economic significance as suggested by the univariate analysis, but precisely estimated, having a partial effect of -0.16. On the other hand, time to maturity at default is of nearly the same economic significance as suggested in exploratory analysis, with a partial effect of -0.013. We interpret this as in line with various well-known seasoning effects in the default prediction literature, where earlier default may be indicative of a weaker credit risk profile, which in turn can imply higher loss severity.

We conclude this section by a discussion of the model diagnostics for the FIML simultaneous equation estimation (Exhibit 3). The broad fit to the data of this model as measured by the likelihood ratio is robust at conventional significance levels. The in-sample R-squared is lower in the instrument equation but almost the same in the obligor equation, 0.66 versus 0.70, respectively. However, out-of-sample this is reversed, with the improvement in the obligor equation but negligible difference in the instrument equation, 0.51 versus 0.45, respectively. This is a little disappointing, as we would expect to see improvement for instrument-level predictive accuracy on at least an in-sample basis. However, there is evidence of more accurate prediction of instrument LGD in looking at the Hoshmer-Lemeshow (HL) statistics. While for obligor LGD the p-value in-sample is 0.55, out-of-sample it is 0.43.

However, in the discriminatory accuracy measures, in both equations we see robust magnitudes of areas under ROC curves: 0.8920 in-sample versus 0.8876 out-of-sample in the obligor equation; and 0.9010 versus 0.8292, in and out of sample respectively, in the instrument equation. However, the Kolmogorov-Smirnov p-values show good separation, but we note that they are about one tenth of the values in the obligor as compared to the instrument equation in-sample.

Discussion and Conclusions

In this study we empirically investigate ultimate loss given default by studying its determinants. In the process, we build a predictive econometric model and evaluate the rank ordering and predictive accuracy properties of this model. We use a sample of 871 large corporate bankruptcies and out-of-court settlements on firms, rated by Moody's in the period of 1985–2008, for which the complete capital structures are available. We contribute to the literature on LGD along several dimensions: understanding it at different levels (obligor versus instrument), finding a comprehensive set of determinants, and evaluating our econometric models rigorously according to well-accepted measures of model performance on out-of-sample basis. Further, we have tried to propose a framework that would prove useful to traders of distressed debt, risk managers, and supervisors. Our main contribution to the literature is along two broad dimensions. First, we construct a robust econometric model that can coherently model the theoretically motivated interplay between LGD at the obligor and instruments levels. Second, we conduct an empirical quest for a set of determinants of ultimate LGD, motivated by both corporate finance theories and risk management practices.

The contribution along the dimension of model development is manifest in several stages. First, we implement an improved econometric methodology by analyzing the ultimate LGD through estimating a model in the BLGLM class, which has some desirable properties relative to alternative approaches (e.g., approximate linear regression model of a normal transformed LGD), including better overall model performance and quality of coefficient estimates. We focus on the obligor-level LGD, similarly to Carey and Gordy [2007], building upon their intuition that facility level recoveries can be likened to collar options on the estate level recovery. This involves extending prior work by modeling the ultimate loss given default jointly at the firm and instrument level. To implement this econometrically, we build a simultaneous equation (full information, maximum likelihood—FIML) version of BLGLM. Finally, the model is validated rigorously on an out-of-time and out-of-sample framework.

The next contribution is analysis of new as well as previously considered explanatory variables in a unified framework. We confirm many of the stylized facts and findings of the literature in regard to the determinants of the ultimate LGD, and find in addition the independent significance of macroeconomic factors, industry conditions, equity returns, and the price of traded debt at default. We demonstrate the statistical and economic significance of debt and equity market determinants of LGD: price of instrument debt at default in the instrument LGD equation, the principal weighted obligor LGD at default, and cumulative abnormal returns on equity prior to default, the latter two in the obligor LGD equation. The economic implication is that there may be information embedded in the market beyond that distilled from the traditional style of credit analysis that workout specialists or vulture investors may undertake. That is, for defaulted bonds and loans of companies that have either traded debt or equity, incorporation of market signals may improve forecasts of expected recoveries. We also find that firms having been investment grade at origination have significantly lower ultimate LGDs, consistent with recent research (Cantor et al. [2007]). We interpret this as evidence that ratings assessments may contain information on not only default likelihood, but more broadly some notion of expected credit loss, which embeds losses in the event of default.

Regarding macroeconomic and industry-wide effects, we find both the aggregate default rate (Moody's trailing 12-month speculative) and a measure of the economic growth (S&P 500 returns) to significantly influence the ultimate LGD through the impact on the obligor equation. Over and above this, we find that dummy variables for the 2000–2002 and the 1991–1992 recessionary episodes to significantly influence LGD at the obligor level, consistent with evidence presented by Carey and Gordy [2007]. Considering industry-specific factors, we find industry profit margin and dummy variables for the technology or utility industries to be significant, consistent with Acharya et al. [2007] and evidence of the independent importance of a separate systematic factor governing LGD apart from that driving default (Frye [2000a, 2000b, 2000c]). These findings contribute to our understanding of the degree of systematic risk present in this asset class of defaulted fixed income instruments; there are also implications for the Basel II regulatory capital regulations, in which financial institutions must assess the degree to which LGD on their portfolios may be elevated in periods of economic downturn.

We also demonstrate the economic and statistical significance of firm-specific effects, as measured by financial variables in the obligor equation. In particular, measures of firm size, leverage, tangibility, market valuation, cash flow, and liquidity are all found to significantly and inversely influence the ultimate LGD. Furthermore, we document a new finding, that larger firms have significantly lower LGDs and larger loans significantly higher LGDs. The economic import of this lies in the quantification of idiosyncratic risk associated with this asset class and the utility of such factors in helping market participants better inform their expectations.

We further find that contractual features are economically and statistically significant determinants of facility LGD: a better ranking of collateral, less debt above/more debt below, and relative seniority of creditor class are all associated with lower LGD. This confirms findings of Keisman and van de Castle [2000] with regard to the importance of loan structure in determining recoveries, which again speaks to an augmented understanding of the idiosyncratic components of recoveries. Finally, we also find that capital structure variables exert particular influences on LGD: The number of creditor classes, and proportions of secured and bank debt, are all significantly and inversely related to the ultimate LGD. The latter finding is consistent with the finding by Carey and Gordy [2007], while the first two are new to the literature, and the first result is the opposite of the finding in Acharya et al. [2007]. The result on debt composition is important from the finance perspective of better understanding the role of financial institutions in contributing to the efficiency of markets through their role in optimally monitoring lenders in the face of potential informational imbalances.

Exhibit 1
Characteristics of loss given default observations by default type: Instrument versus obligor level analysis (Moody's Rated Defaults 1985–2008)

Panel 1 - Instrument Level Observations

		LGD at Default	Discounted LGD	Number of Creditor Classes	Principal at Default ($1,000)	Time-to-Resolution (years)
Bankruptcy	Count (Total)	996	3273			
	Average	62.89%	48.38%	2.42	139,012	1.60
	Median	69.62%	56.80%	2.00	80,386	1.30
	Std.Dev.	28.42%	39.96%	0.89	222,897	1.21
	Minimum	-12.00%	-124.19%	1.00	10	0.00
	Maximum	99.80%	100.00%	6.00	4,600,000	9.18
Out-of-Court	Count (Total)	122	629			
	Average	45.92%	16.31%	2.55	160,241	0.21
	Median	65.44%	39.25%	2.00	63,966	0.96
	Std.Dev.	26.96%	30.63%	0.98	288,905	0.55
	Minimum	-7.87%	-99.35%	1.00	10	0.00
	Maximum	98.00%	100.00%	6.00	3,000,000	5.64
Total	Count (Total)	1118	3902			
	Average	61.04%	43.21%	2.44	142,434	1.38
	Median	68.41%	46.09%	2.00	76,433	1.09
	Std.Dev.	28.74%	40.37%	0.91	234,883	1.24
	Minimum	-12.00%	-124.19%	1.00	10	0.00
	Maximum	99.80%	100.00%	6.00	4,600,000	9.18

Panel 2 - Obligor Level Observations

		LGD at Default	Discounted LGD	Number of Creditor Classes	Principal at Default ($1,000)	Time-to-Resolution (years)
Bankruptcy	Count (Total)	392	728			
	Average	65.10%	51.41%	2.18	624,849	1.55
	Median	68.69%	55.74%	2.00	262,778	1.33
	Std.Dev.	23.44%	30.43%	0.88	1,559,892	1.08
	Minimum	-5.52%	-89.65%	1.00	1,979	0.06
	Maximum	99.00%	100.00%	6.00	32,279,012	9.18
Out-of-Court	Count (Total)	68	143			
	Average	54.06%	23.80%	2.31	704,836	0.42
	Median	66.45%	21.38%	2.00	246,655	0.08
	Std.Dev.	24.60%	28.45%	0.91	2,315,345	0.75
	Minimum	5.29%	-64.89%	1.00	14,495	0.00
	Maximum	97.47%	94.01%	6.00	19,177,444	5.64
Total	Count (Total)	460	871			
	Average	63.46%	46.88%	2.20	637,981	1.36
	Median	68.22%	50.44%	2.00	261,291	1.14
	Std.Dev.	23.91%	31.79%	0.88	1,705,631	1.11
	Minimum	-5.52%	-89.65%	1.00	1,979	0.00
	Maximum	99.00%	100.00%	6.00	32,279,012	9.18

Note. LGD at Default = 1 minus the price of defaulted debt at the time of default; for obligors, weighted by the amounts outstanding at default. Discounted LGD = The ultimate dollar loss given default on the defaulted debt instrument: 1 minus recovery at emergence from bankruptcy or time of final settlement as a percent of par; alternatively, this can be expressed as (amount outstanding at default total ultimate dollar recovery) / (amount outstanding at default); for obligors, weighted by the amounts outstanding at default. Number of Creditor Classes = Major creditor classes as defined by the bankruptcy court or by mutual agreement in the out-of-court settlement. Principal at Default = The total instrument or obligor outstanding at default. Time to Resolution = The time in years from the (first) instrument default date to the time of ultimate recovery for instruments (obligors).

Exhibit 2

Summary statistics on selected variables and correlations with loss given default (Moody's Rated Defaults 1985–2008)

Category	Variable	Count	Median	Standard Deviation	Spearman Rank Correlation with Obligor LGD	Spearman Rank Correlation with Instrument LGD
Financial Variables	Leverage Ratio (Book Value)	628	95.08%	64.89%	-6.96%	-1.34%
	Debt to Equity Ratio (Market)	471	87.05%	18.56%	-7.79%	-10.73%
	Book Value of Assets	628	454.95	1,566.69	-9.01%	10.15%
	Market Value of Equity	470	1.7918	0.8413	-7.48%	11.12%
	Tobin's Q	427	0.7494	0.7124	8.27%	13.31%
	Intangibles Ratio	540	10.27%	19.69%	14.17%	2.45%
	Working Capital / Total Assets	576	3.66%	45.23%	-4.54%	-7.10%
	Operating Cash Flow	590	1.3400	482.58	-4.29%	-10.58%
	Return on Assets - Industry	610	-8.78%	26.37%	-11.97%	-36.35%
	Profit Margin - Industry	743	1.29%	7.30%	-22.86%	-13.79%
Capital Structure	Number of Instruments	871	3.0000	5.0013	-1.64%	10.86%
	Number of Creditor Classes	871	2.0000	0.8828	-5.58%	-2.68%
	Percent Secured Debt	871	0.3767	0.3448	-19.78%	-4.95%
	Percent Bank Debt	871	0.2717	0.2970	-23.48%	-2.19%
	Percent Subordinated Debt	871	0.1935	0.3671	13.75%	7.48%
Credit Quality / Market	Number of Downgrades	520	1.0000	1.6403	-15.69%	-11.02%
	LGD at Default - Obligor	657	0.6822	0.2391	49.08%	33.70%
	Credit Spread - Instrument	3838	0.0750	0.0860	4.21%	19.37%
	LGD at Default - Instrument	1597	0.6841	0.2874	46.96%	66.69%
	Cumulative Abnormal Returns	286	-0.1222	0.5648	-19.60%	-26.86%
Contractua l	Seniority Rank	3902	1.0000	0.7952	-4.13%	37.31%
	Collateral Rank	3902	6.0000	2.5353	7.21%	47.48%
	Percent Debt Below	3902	8.15%	29.34%	-0.88%	-48.93%
	Percent Debt Above	3902	0.00%	29.38%	-5.14%	38.70%
Vintag e	Time Between Defaults	3902	0.0000	0.3871	-5.80%	-12.53%
	Time-to-Maturity	3902	4.5096	4.6734	27.03%	27.06%
Macro	Moody's Speculative Default Rate	871	0.0639	0.0312	1.76%	4.53%
	S&P 500 Return	871	0.0096	0.0141	-7.29%	-12.50%

Note. This table presents distributional statistics and spearman rank order correlations with LGD (both at the obligor and instrument levels) for selected variables in our database.

Exhibit 3

Analysis of simultaneous equation modeling of discounted instrument and obligor Loss given default: Full information maximum likelihood estimation using beta link generalized linear model (Moody's Rated Defaults 1985–2008)

Category	Variable	Instrument Partial Effect	Instrument P-Value	Obligor Partial Effect	Obligor P-Value
Financial	Debt to Equity Ratio (Market)			-0.0903	2.55E-03
	Book Value			-0.0814	0.0174
	Tobin's Q			0.0729	8.73E-03
	Intangibles Ratio			0.0978	7.02E-03
	Working Capital / Total Assets			-0.1347	4.54E-03
	Operating Cash Flow			-8.31E-03	0.0193
Industry	Profit Margin - Industry			-0.0917	1.20E-03
	Industry - Utility			-0.1506	8.18E-03
	Industry - Technology			0.0608	2.03E-03
Contractual	Senior Secured	0.0432	0.0482		
	Senior Unsecured	0.0725	3.11E-03		
	Senior Subordinated	0.2266	1.21E-03		
	Junior Subordinated	0.1088	0.0303		
	Collateral Rank	0.1504	4.26E-12		
	Percent Debt Above	0.1241	3.84E-03		
	Percent Debt Below	-0.2930	7.65E-06		
Time	Time Between Defaults			-0.1853	7.40E-04
	Time-to-Maturity	0.0255	0.0084		
Capital Structure	Number of Creditor Classes			-0.0975	1.20E-03
	Percent Secured Debt			-0.1403	7.56E-03
	Percent Bank Debt			-0.2382	7.45E-03
Credit Quality / Market	Investment Grade at Origination			-0.0720	4.81E-03
	Principal at Default	8.99E-03	1.14E-03		
	Cumulative Abnormal Returns			-0.2753	1.76E-04
	Ultimate LGD - Obligor	0.5643	7.82E-06		
	LGD at Default - Obligor			0.1906	4.05E-04
	LGD at Default - Instrument	0.2146	1.18E-14		
Legal	Prepackaged Bankruptcy			-0.0406	5.38E-03
	Bankruptcy Filing			0.1429	5.00E-03
Macro	1989-1991 Recession			0.0678	0.0474
	2000-2002 Recession			0.1074	0.0103
	Moody's Speculative Default Rate			0.0726	1.72E-04
	S&P 500 Return			-0.1392	2.88E-04
Diagnostics		In-Smpl	Out-Smpl	In-Smpl	Out-Smpl
	Number of Observations	568	114	568	114
	Log-Likelihood	1.72E-10	9.60E-08	1.72E-10	9.60E-08
	Pseudo R-Squared	0.6997	0.6119	0.5822	0.4744
	Hoshmer-Lemeshow	0.4115	0.3345	0.5204	0.3907
	Area under ROC Curve	0.8936	0.7653	0.8983	0.7860
	Kolmogorov-Smirnov	1.12E-07	4.89E-06	1.42E-07	6.87E-06

Note. This table presents results from the full information maximum likelihood estimation (FIMLE) of the beta link generalized linear model (BLGLM) model for firm- and instrument-level LGD. FIMLE is used to model the endogeneity of the relationship between LGD at the firm and instrument levels in an internally consistent manner.

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